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Bounds for symmetric elliptic integrals

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Abstract

Lower and upper bounds for the four standard incomplete symmetric elliptic integrals are obtained. The bounding functions are expressed in terms of the elementary transcendental functions. Sharp bounds for the ratio of the complete elliptic integrals of the second kind and the first kind are also derived. These results can be used to obtain bounds for the product of these integrals. It is shown that an iterative numerical algorithm for computing the ratios and products of complete integrals has the second order of convergence.

Keywords: Standard elliptic integrals; The R-hypergeometric functions; Inequalities; Means

1. Introduction

In some problems that arise in science and engineering one has to deal often with the elliptic integrals. A classical result, which is due to Abel, states that they cannot be represented by the elementary transcendental functions. All integrals discussed in this paper are the standard elliptic integrals. They are homogeneous functions of two or three or four variables and they simplify to Legendre's elliptic integrals for special values of their variables. Bounds for the latter class of elliptic integrals can be found in [1–3,13].

The goal of this paper is to derive bounds for four incomplete elliptic integrals with the bounding functions being some elementary transcendental functions. These results are presented in Section 3. Bounds for the ratio of the complete integrals of

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the first and second kinds are obtained in Section 4. These results can be used to obtain bounds for the product of these integrals.

In what follows, we will assume that x, y, z are nonnegative numbers and that at most one of them is 0. The symmetric integral of the first kind is defined as

$$R_F(x,y,z) = \frac{1}{2}\int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2}\,dt \tag{1.1}$$

(see, e.g., [7,8,10]). Clearly R_F is symmetric and homogeneous of degree $-\frac{1}{2}$ in x, y, z and satisfies $R_F(x,x,x) = x^{-1/2}$.

Let $p > 0$. Elliptic integral of the third kind,

$$R_J(x,y,z,p) = \frac{3}{2}\int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2}(t+p)^{-1}\,dt \tag{1.2}$$

is symmetric in x, y, z, homogeneous of degree $-\frac{3}{2}$ in x, y, z, p and satisfies $R_J(x,x,x,x) = x^{-3/2}$ (see [8,10]). A degenerate case of R_J is the elliptic integral of the second kind

$$\begin{aligned} R_D(x,y,z) &= R_J(x,y,z,z) \\ &= \frac{3}{2}\int_0^\infty [(t+x)(t+y)]^{-1/2}(t+z)^{-3/2}\,dt \end{aligned} \tag{1.3}$$

which is symmetric in x and y only. A completely symmetric integral of the second kind

$$\begin{aligned} &R_G(x,y,z) \\ &= \frac{1}{4}\int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2}\left(\frac{x}{t+x}+\frac{y}{t+y}+\frac{z}{t+z}\right)t\,dt \end{aligned} \tag{1.4}$$

is symmetric and homogeneous of degree $\frac{1}{2}$ in its variables, satisfies $R_G(x,x,x) = x^{1/2}$, and is well defined if any or all of x, y, z are 0 (see [7,8,10]). All four integrals defined above are the incomplete integrals. Two complete integrals, of the first kind and the second kind, are defined as follows:

$$R_K(x,y) = \frac{2}{\pi}R_F(x,y,0) = \frac{1}{\pi}\int_0^\infty [(t+x)(t+y)]^{-1/2}\,dt \tag{1.5}$$

and

$$\begin{aligned} R_E(x,y) &= \frac{4}{\pi}R_G(x,y,0) \\ &= \frac{1}{\pi}\int_0^\infty [(t+x)(t+y)]^{-1/2}\left(\frac{x}{t+x}+\frac{y}{t+y}\right)t\,dt \end{aligned} \tag{1.6}$$

(see [7,8,10]).

An important elementary transcendental function used in this paper, denoted by R_C, is the degenerate case of R_F,

$$R_C(x,y) = R_F(x,y,y) = \frac{1}{2}\int_0^\infty (t+x)^{-1/2}(t+y)^{-1}\,dt \tag{1.7}$$

($x \geqslant 0$, $y > 0$). It is known that

$$R_C(x,y) = \begin{cases} (y-x)^{-1/2} \arccos(x/y)^{1/2}, & x<y, \\ (x-y)^{-1/2} \operatorname{arccosh}(x/y)^{1/2}, & x>y \end{cases} \tag{1.8}$$

(see [7, (6.9–15);10]). Let us note that $R_C(0,y) = \pi/(2y^{1/2})$.

2. The R-hypergeometric functions

All elliptic integrals defined in Section 1 can be represented by the R-hypergeometric functions. For the reader's convenience we give below a definition of this important class of special functions. In what follows, we will employ notation and definitions introduced in Carlson's monograph [7]. The symbols $\mathbb{R}_+$ and $\mathbb{R}_>$ will stand for the nonnegative semi-axis and the set of positive numbers, respectively. For $b = (b_1, \dots, b_n) \in \mathbb{R}^n_+$ and $X = (x_1, \dots, x_n) \in \mathbb{R}^n_>$ the R-hypergeometric function of order $a \in \mathbb{R}$ with the parameters b and variables X is defined as

$$R_{-a}(b;X) = \int_{E_{n-1}} (u \cdot X)^{-a} \, d\mu_b(u), \tag{2.1}$$

where

$$E_{n-1} = \{(u_1, \dots, u_{n-1}) \colon u_i \geqslant 0, \ 1 \leqslant i \leqslant n-1, \ u_1 + \cdots + u_{n-1} \leqslant 1\}$$

is the Euclidean simplex, $u = (u_1, \dots, u_{n-1}, u_n)$ where $u_n = 1 - u_1 - \cdots - u_{n-1}$, $u \cdot X = u_1 x_1 + \cdots + u_n x_n$ is the dot product of u and X,

$$d\mu_b(u) = \frac{1}{B(b)} \prod_{i=1}^{n} u_i^{b_i - 1} \, du$$

is the Dirichlet measure on E_{n-1}, B stands for the multivariate beta function and $du = du_1 \dots du_{n-1}$. Function R_{-a} is also called the Dirichlet average of the power function t^{-a}.

Some elementary properties of R_{-a} are listed below:

(i) A vanishing b-parameter can be omitted along with the corresponding variable.
(ii) Permutation symmetry (symmetry in indices $1, \dots, n$ which label the b-parameters and the variables).
(iii) Equal variables can be replaced by a single variable if the corresponding parameters are replaced by their sum. In particular, if all variables are equal, then $R_{-a}(x, \dots, x) = x^{-a}$.

If $a > 0$, then the R-hypergeometric function R_{-a} admits another integral representation [7, (6.8-6)]

$$R_{-a}(b;X) = \frac{1}{B(a,a')} \int_0^\infty t^{a'-1} \prod_{i=1}^{n} (t + x_i)^{-b_i} \, dt, \tag{2.2}$$

where $a' = b_1 + \cdots + b_n - a > 0$.

All six elliptic integrals defined in the previous section can be represented by the R-hypergeometric functions. We have

$$R_F(x,y,z) = R_{-1/2}(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};x,y,z),$$

$$R_J(x,y,z,p) = R_{-3/2}(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},1;x,y,z,p),$$

$$R_D(x,y,z) = R_{-3/2}(\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2};x,y,z),$$

$$R_G(x,y,z) = R_{1/2}(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};x,y,z),$$

$$R_K(x,y) = R_{-1/2}(\tfrac{1}{2},\tfrac{1}{2};x,y),$$

$$R_E(x,y) = R_{1/2}(\tfrac{1}{2},\tfrac{1}{2};x,y) \tag{2.3}$$

(see [7,8]). First formula in (2.3) together with (1.7) give the known result

$$R_C(x,y) = R_{-1/2}(\tfrac{1}{2},1;x,y). \tag{2.4}$$

We close this section with three lemmas which will be used in Section 3.

Lemma 2.1. *Let* $x \geqslant 0$, $y > 0$, $z > 0$ *and let*

$$j(x,y,z) = \frac{3}{2}\int_0^\infty (t+x)^{-1/2}[(t+y)(t+z)]^{-1}\,dt. \tag{2.5}$$

Then

$$j(x,y,z) = R_{-3/2}(\tfrac{1}{2},1,1;x,y,z) = R_J(x,y,y,z) \tag{2.6}$$

and

$$j(x,y,z) = \begin{cases} 3\,\dfrac{R_C(x,y) - R_C(x,z)}{z-y}, & y \neq z, \\ 3\,\dfrac{x^{1/2} - yR_C(x,y)}{2(x-y)y}, & x \neq y = z, \\ x^{-3/2} & x = y = z. \end{cases} \tag{2.7}$$

Proof. For the proof of (2.6) we use formulas (2.2) and (2.1) with $a = 3/2$, $b = (1/2,1,1)$ and $X = (x,y,z)$ and property (iii). We shall establish now (2.7). If $y \neq z$ the first equation follows from (2.5) by partial-fraction decomposition and (1.7). Assume now that $x \neq y = z$. It follows from (2.6) that $j(x,y,y) = R_{-3/2}(\tfrac{1}{2},2;x,y)$. Next to the last line of Table 8.5-1 in [7] gives the second part of (2.7). The third part of (2.7) is an obvious consequence of (2.6) and property (iii). □

For later use we define two functions $d(x,y)$ and $g(x,y)$, where

$$d(x,y) = j(x,x,y) \tag{2.8}$$

($x > 0$, $y > 0$) and

$$g(x,y) = R_{1/2}(\tfrac{1}{2},1;x,y) = R_G(x,y,y) \tag{2.9}$$

($x \geqslant 0$, $y > 0$). It follows from (2.7) that

$$d(x,y) = \begin{cases} 3\dfrac{R_C(x,y) - x^{-1/2}}{x-y}, & x \neq y, \\ x^{-3/2}, & x = y. \end{cases} \tag{2.10}$$

Function g can also be represented by the R-hypergeometric function R_C. First entry of Table 8.5-1 in [7] gives

$$g(x,y) = \begin{cases} \dfrac{x^{1/2} + yR_C(x,y)}{2}, & x \neq y, \\ x^{1/2}, & x = y. \end{cases} \tag{2.11}$$

Inequalities for the R-hypergeometric functions are contained in the next two lemmas.

Lemma 2.2. *Let $b = (b_1, b_2, b_3) \in \mathbb{R}^3_>$, $X = (x, y, z) \in \mathbb{R}^3_>$ and assume that $\min(X) < \max(X)$. Also, let $\lambda = b_1/(b_1 + b_2)$, $\mu = b_2/(b_1 + b_2)$. If $0 < t < 1$, then*

$$\begin{aligned} &\lambda R_t(b_1 + b_2, b_3; x, z) + \mu R_t(b_1 + b_2, b_3; y, z) < R_t(b; X) \\ &\quad < R_t(b_1 + b_2, b_3; \lambda x + \mu y, z). \end{aligned} \tag{2.12}$$

Inequalities in (2.12) *are reversed if either $t > 1$ or $t < 0$ and they become equalities if $t = 0$ or $t = 1$ or $x = y$.*

Proof. This follows immediately from Theorem 3 in [11, (4.18)]. □

Lemma 2.3. *Let $b \in \mathbb{R}^n_>$, $X \in \mathbb{R}^n_>$ and assume that $\min(X) < \max(X)$. Then the following inequalities*

$$R_{-a}(b; X)R_a(b; X) > 1 \quad \text{and} \quad R_{-a}(b; X) + R_a(b; X) > 2 \tag{2.13}$$

hold true for any $a \neq 0$.

Proof. It is known that the R-hypergeometric function R_t is strictly log-convex in t (see [4, Theorem 4]; [7, Appendix B]). Thus

$$R_p^{\lambda} R_q^{1-\lambda} > R_{\lambda p + (1-\lambda)q}$$

($pq \neq 0$, $0 \leqslant \lambda \leqslant 1$). Letting above $-p = q = a$ and $\lambda = 1/2$, we obtain

$$R_{-a}R_a > R_0^2 = 1.$$

The second inequality in (2.13) follows from the first one and the arithmetic mean–geometric mean inequality. □

It follows from (2.13) and (2.3) that

$$R_F R_G > 1 \quad \text{and} \quad R_F + R_G > 2 \tag{2.14}$$

and

$$R_K R_E > 1 \quad \text{and} \quad R_E + R_K > 2. \tag{2.15}$$

Inequalities (2.15) also follow from Theorem 1.1 in [2].

3. Bounds for the incomplete elliptic integrals

Lower and upper bounds for the four elliptic integrals R_F, R_J, R_D, and R_G are derived in this section.

In what follows, the letters α and β will stand for the roots of the Chebyshev polynomial $T_2(t) = 8t^2 - 8t + 1$ on $[0, 1]$, i.e., $\alpha = (1 - 1/\sqrt{2})/2$ and $\beta = (1 + 1/\sqrt{2})/2$. The following inequalities will be used in the proof of the main result of this section.

Lemma 3.1. *Let $x > 0$, $y > 0$, $A = (x + y)/2$ and let $u = \alpha x + \beta y$ and $v = \alpha y + \beta x$. Then the following inequalities*

$$\begin{aligned} \frac{1}{2}\left(\frac{1}{t+u} + \frac{1}{t+v}\right) &\leqslant [(t+x)(t+y)]^{-1/2} \\ &\leqslant \frac{1}{2}\left[\frac{1}{t+A} + \frac{1}{2}\left(\frac{1}{t+x} + \frac{1}{t+y}\right)\right] \end{aligned} \tag{3.1}$$

are satisfied for all $t \in \mathbb{R}_+$.

Proof. By the inequality of arithmetic and geometric means of a^2 and g^2 we have

$$\frac{2ag}{a^2 + g^2} \leqslant 1 \leqslant \frac{a^2 + g^2}{2ag}.$$

Dividing all three terms by g, and letting a and g be the arithmetic and geometric means, respectively, of $t + x$ and $t + y$ we obtain

$$\begin{aligned} \frac{2(t+A)}{(t+A)^2 + (t+x)(t+y)} &\leqslant [(t+x)(t+y)]^{-1/2} \\ &\leqslant \frac{(t+A)^2 + (t+x)(t+y)}{2(t+A)(t+x)(t+y)}. \end{aligned} \tag{3.2}$$

Partial-fraction decomposition of the rational functions in (3.2) gives the desired result. $\square$

Let us note that the first and third members of (3.1) provide shape preserving approximations to the function $[(t + x)(t + y)]^{-1/2}$, i.e., they share monotonicity and convexity properties of the function to be approximated.

We are now in a position to state and prove the main result of this section.

Theorem 3.2. *Let x, y, z and p be nonnegative numbers and let the symbols u, v, and A have the same meaning as in Lemma 3.1. Then*

$$\begin{aligned}&\tfrac{1}{2}[R_C(z,u)+R_C(z,v)]\\&\quad\leqslant R_F(x,y,z)\\&\quad\leqslant \tfrac{1}{2}[R_C(z,A)+\tfrac{1}{2}(R_C(z,x)+R_C(z,y))]\quad (x>0,\ y>0),\end{aligned}\tag{3.3}$$

$$\begin{aligned}&\tfrac{1}{2}[j(z,p,u)+j(z,p,v)]\\&\quad\leqslant R_J(x,y,z,p)\\&\quad\leqslant \tfrac{1}{2}[j(z,p,A)+\tfrac{1}{2}(j(z,p,x)+j(z,p,y))]\quad (x>0,y>0,p>0)\end{aligned}\tag{3.4}$$

and

$$\tfrac{1}{2}[g(z,x)+g(z,y)]\leqslant R_G(x,y,z)\leqslant g(z,A)\quad (x>0,y>0).\tag{3.5}$$

Inequalities (3.3)–(3.5) *become equalities if either $x=y$ or $x=y=z$ or $x=y=z=p$ where the latter condition applies to* (3.4) *only.*

Proof. For the proof of (3.3) we multiply all terms in (3.1) by $(1/2)(t+z)^{-1/2}$and next integrate, from 0 to infinity, all members of the resulting inequality. Application of (1.1) and (1.7) completes the proof. Inequalities (3.4) are derived from (3.1) in an analogous manner. First, we multiply all members by $(3/2)(t+z)^{-1/2}(t+p)^{-1}$ and next use formulas (1.2) and (2.5). Inequalities (3.5) follow from (2.12) by letting $t=b_1=b_2=b_3=1/2$. The fourth formula in (2.3) is used together with (2.9) and the permutation symmetry (ii), to obtain the desired result. The last statement of the theorem follows from the fact that the first and third members in (3.3)–(3.5) are equal in the stated cases. □

Corollary 3.3. *Let $x>0$, $y>0$, and $z>0$. Then*

$$\begin{aligned}\tfrac{1}{2}[d(z+u)+d(z,v)]&\leqslant R_D(x,y,z)\\&\leqslant \tfrac{1}{2}[d(z,A)+\tfrac{1}{2}(d(z,x)+d(z,y))].\end{aligned}\tag{3.6}$$

Equalities hold in (3.6) *only if $x=y$ or $x=y=z$.*

Proof. This result follows immediately from (1.3), (3.4), and (2.8). □

New bounds for the function R_C have been established in [12, Theorem 3.3]:

$$3/(p_n+2q_n)\leqslant R_C(x,y)\leqslant (p_n q_n^2)^{-1/3}\tag{3.7}$$

($x\geqslant 0$, $y>0$, $n=0,1,\ldots$), where the sequences $\{p_n\}_0^\infty$ and $\{q_n\}_0^\infty$ are defined as $p_0=\sqrt{x}$, $q_0=\sqrt{y}$, $p_{n+1}=(p_n+q_n)/2$ and $q_{n+1}=(p_{n+1}q_n)^{1/2}$ $(n\geqslant 0)$. Inequalities (3.7) together with the results of Theorem 3.2 provide weaker bounds for the integrals in question. We omit further details.

The following result

$$R_C(z, A) \leqslant R_F(x, y, z) \leqslant R_C(z, \sqrt{xy}) \tag{3.8}$$

is known (see [10]). We shall show that the lower bound in (3.8) is weaker than the corresponding bound in (3.3). To this aim we use the fact that the function $R_C(\cdot, y)$ is a convex function in y. This in turn implies that

$$\tfrac{1}{2}[R_C(z, u) + R_C(z, v)] \geqslant R_C\left(z, \frac{u+v}{2}\right) = R_C(z, A).$$

More bounds for the incomplete symmetric elliptic integrals can be found in [5,6,9,10].

4. Bounds for the ratio R_E/R_K

The goal of this section is to derive simple bounds for the ratio $R_E(x, y)/R_K(x, y)$ $(x>0,\ y>0)$. For later use, we recall the definition of the celebrated Gauss arithmetic–geometric mean $AGM \equiv AGM(x, y)$ of x and y. For $n = 0, 1, \ldots$ let

$$a_0 = x, \quad b_0 = y, \quad a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}. \tag{4.1}$$

It is well known that

(a) $b_n \leqslant b_{n+1} \leqslant \cdots < AGM < \cdots \leqslant a_{n+1} \leqslant a_n \ (n \geqslant 1)$,
(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = AGM$,
(c) $R_K(x, y) AGM(\sqrt{x}, \sqrt{y}) = 1$,

(see, e.g., [7, Ex. 6.10-8]).

We need the following.

Lemma 4.1. *The following inequalities*

$$\sqrt{xy} \leqslant \frac{R_E(x, y)}{R_K(x, y)} \leqslant \left(\frac{\sqrt{x} + \sqrt{y}}{2}\right)^2 \tag{4.2}$$

hold true.

Proof. We multiply (c) by R_E and use (2.15) and (a) to prove that

$$R_E(x, y) \geqslant AGM(\sqrt{x}, \sqrt{y}) \geqslant (xy)^{1/4}. \tag{4.3}$$

Application of (c) to (4.3) gives the first inequality in (4.2). For the proof of the second inequality in (4.2) we use the following result [7, Problem 9.5-4]

$$\frac{R_E(x^2, y^2)}{R_K(x^2, y^2)} = a_1^2 - \sum_{n=1}^{\infty} 2^n (a_{n+1}^2 - b_{n+1}^2) \leqslant a_1^2 = \left(\frac{x+y}{2}\right)^2,$$

where the sequences $\{a_n\}_0^\infty$ and $\{b_n\}_0^\infty$ are defined in (4.1). Replacing x by $x^{1/2}$ and y by $y^{1/2}$ we obtain the desired result. $\square$

Before we state and prove the main result of this section let us introduce more notation. Two sequences $\{x_n\}_0^\infty$ and $\{y_n\}_0^\infty$ are defined recursively as follows:

$$x_0 = x, \quad y_0 = y, \quad x_{n+1} = \left(\frac{\sqrt{x_n} + \sqrt{y_n}}{2}\right)^2, \quad y_{n+1} = \sqrt{x_n y_n}, \tag{4.4}$$

$n = 0, 1, \ldots$. Since these sequences are related to (4.1) by $x_n = a_n^2$ and $y_n = b_n^2$, it follows that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = AGM(\sqrt{x}, \sqrt{y})^2.$$

Also, we will need another pair of sequences, denoted by $\{u_n\}_0^\infty$ and $\{v_n\}_0^\infty$, where

$$u_n = 2^n x_{n+1} - \delta_n, \quad v_n = 2^n y_{n+1} - \delta_n \tag{4.5}$$

and

$$\delta_n = \sum_{k=1}^{n} 2^{k-1} y_k, \tag{4.6}$$

$n = 0, 1, \ldots$.

The main result of this section, generalizing (4.2) (the case $n = 0$) and Corollary 4.3 (the case $n = 1$), reads as follows

Theorem 4.2. *Let* $n = 0, 1, \ldots$. *Then*

$$v_n \leqslant \frac{R_E(x, y)}{R_K(x, y)} \leqslant u_n, \tag{4.7}$$

where the lower and upper bounds in (4.7) *are monotonic in the following sense*:

$$v_{n-1} \leqslant v_n \leqslant u_n \leqslant u_{n-1} \tag{4.8}$$

$(n = 1, 2, \ldots)$.

Proof. We use the Landen transformation

$$R_E(x, y) = 2R_E\left[\left(\frac{\sqrt{x} + \sqrt{y}}{2}\right)^2, \sqrt{xy}\right] - \sqrt{xy}\, R_K(x, y)$$

(see [7, Problem 9.5-2]) together with the quadratic transformation for R_K

$$R_K(x, y) = R_K\left[\left(\frac{\sqrt{x} + \sqrt{y}}{2}\right)^2, \sqrt{xy}\right]$$

(see [7, (6.10-5)]) and (4.4) to obtain

$$\frac{R_E(x, y)}{R_K(x, y)} = 2\frac{R_E(x_1, y_1)}{R_K(x_1, y_1)} - y_1. \tag{4.9}$$

Repeated application of (4.9) implies the following result:

$$\frac{R_E(x, y)}{R_K(x, y)} = 2^n \frac{R_E(x_n, y_n)}{R_K(x_n, y_n)} - \delta_n$$

($n = 0, 1, \ldots$). Application of (4.2) to the first term on the right-hand side of the last formula gives, in conjunction with (4.4), the desired result (4.7). For the proof (4.8) we rewrite formulas (4.5) as

$$u_n = u_{n-1} + 2^n\left(x_{n+1} - \frac{x_n + y_n}{2}\right) \tag{4.10}$$

and

$$v_n = v_{n-1} + 2^n(y_{n+1} - y_n) \tag{4.11}$$

($n = 1, 2, \ldots;\ u_0 = ((\sqrt{x} + \sqrt{y})/2)^2,\ v_0 = \sqrt{xy}$). The first inequality in (4.8) follows from (4.11) because $y_n \leqslant y_{n+1}$ and the third inequality in (4.8) is a consequence of

$$x_{n+1} = \left(\frac{\sqrt{x_n} + \sqrt{y_n}}{2}\right)^2 \leqslant \frac{x_n + y_n}{2}.$$

The proof is complete. □

Corollary 4.3. *Let $A = (x + y)/2$ and $G = \sqrt{xy}$. Then*

$$2\sqrt{\frac{A+G}{2}G} - G \leqslant \frac{R_E(x,y)}{R_K(x,y)} \leqslant \sqrt{\frac{A+G}{2}G} + \frac{1}{4}(A - G).$$

Proof. This follows from (4.7) and (4.10)–(4.11) by letting $n = 1$. □

Let us note that the sequence $\{u_n - v_n\}_0^\infty$ converges quadratically to 0 as $n \to \infty$. This is a consequence of the following formula:

$$\frac{u_n - v_n}{(u_{n-1} - v_{n-1})^2} = \frac{1}{2^{n+2}x_{n+1}} \tag{4.12}$$

($n = 0, 1, \ldots$). For the proof of (4.12) we use (4.4)–(4.6) to obtain

$$\begin{aligned} u_n - v_n &= 2^n(x_{n+1} - y_{n+1}) = 2^n\left(\frac{\sqrt{x_n} - \sqrt{y_n}}{2}\right)^2 = 2^n\,\frac{[2^{n-1}(x_n - y_n)]^2}{[2^n(\sqrt{x_n} + \sqrt{y_n})]^2} \\ &= \frac{(u_{n-1} - v_{n-1})^2}{2^n(\sqrt{x_n} + \sqrt{y_n})^2} = \frac{(u_{n-1} - v_{n-1})^2}{2^{n+2}x_{n+1}}. \end{aligned}$$

Hence (4.12) follows.

We close this section with the remark that the product and the quotient of R_E and R_K satisfy the following relation:

$$R_E(x,y)R_K(x,y) = \frac{R_E(x,y)}{AGM(\sqrt{x}, \sqrt{y})^2 R_K(x,y)}$$

which follows easily from (c).

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